

The Breakup Condition of Shearless KAM Curves in the Quadratic Map

Susumu SHINOHARA* and Yoji AIZAWA

Department of Applied Physics, Waseda University, Tokyo 169

February 5, 2008

Abstract

We determined the exact location of the shearless KAM curve in the quadratic map and numerically investigate the breakup thresholds of those curves in the entire parameter space. The breakup diagram reveals many sharp singularities like fractals on the reconnection thresholds of the twin-chains with rational rotation numbers.

1 Introduction

In this paper we focus on the complicated phenomena which occur in the hamiltonian systems beyond the twist condition. Because of the violation of the twist condition[1, 2], the KAM theorem and the Poincaré-Birkhoff theorem do not hold near the shearless KAM curve[8], where the rotation number takes an extremum value. When a perturbation is added, very complicated phase space structures are observed in the vicinity of the shearless KAM curve: twin resonant chains appear on both sides of the shearless KAM curve, and at a certain parameter value the separatrices of the twin-chains merge. These phenomena, so-called reconnections of twin-chains, are generic ones in many nontwist systems[3]-[8].

The quadratic map, which is the simplest example of a nontwist system, has been studied from the viewpoint of the reconnection of twin-chains, and the breakup mechanism of the shearless KAM curve has been studied to a certain extent[3, 4, 8]. However the detailed phase space structures and the breakup condition of shearless KAM curves in the entire parameter space have not yet been elucidated completely.

In this paper, carrying out with the quadratic map, we derive the exact location of the shearless KAM curve and numerically determine the breakup threshold of shearless KAM curves in the entire parameter space.

The quadratic map is the following two-dimensional area-preserving map $(I_n, \theta_n) \mapsto (I_{n+1}, \theta_{n+1})$,

$$T : \begin{cases} I_{n+1} = I_n - K \sin(\theta_n), \\ \theta_{n+1} = \theta_n + f_\mu(I_{n+1}), \end{cases} \pmod{2\pi} \quad (1)$$

$$f_\mu(I) = 2\pi\mu - I^2,$$

where K represents the strength of the perturbation, and μ is the rotation number of the shearless KAM curve in the integrable case, i.e., $K = 0$. Because the variable θ is 2π -periodic, μ is also periodic with period 1. Thus it is enough for us to consider the parameter region

*E-mail: susumu@aizawa.phys.waseda.ac.jp

$-0.5 \leq \mu < 0.5$.

The quadratic map T can be rewritten as $T = M_2 M_1$ by use of the product of two involutions M_1 and M_2 , where $M_1^2 = M_2^2 = 1$ [4, 8] and,

$$M_1 : \begin{cases} I' = I - K \sin(\theta), \\ \theta' = -\theta, \end{cases} \pmod{2\pi} \quad (2)$$

$$M_2 : \begin{cases} I' = I, \\ \theta' = -\theta + 2\pi\mu - I^2. \end{cases} \pmod{2\pi} \quad (3)$$

Moreover the mapping T has the symmetry S , i.e., $ST = TS$ [8], where

$$S : \begin{cases} I' = -I, \\ \theta' = \theta - \pi. \end{cases} \pmod{2\pi} \quad (4)$$

2 The reconnection of twin-chains

Since the twist function $f_\mu(I)$ is quadratic, the mapping T does not satisfy the twist condition ($df_\mu/dI \neq 0$ for $\forall I$), and the twin resonant chains appear on both sides of a shearless KAM curve [3, 4, 8]. In Figs.1 and 2, period-one and period-two twin-chains are shown for various values of K , with μ fixed. When we change the value of the parameters μ or K , the twin-chains exhibit topologically different structures; Fig.1(a) for $K = 0.05$ shows that two twin-chains with period-one are generated on both sides of the shearless KAM curve, and Fig.1(c) for $K = 0.20$ shows the saddle-node fusion of those two twin-chains. At the critical parameter value $K = 0.1173906615$, the separatrices of these twin-chains merge in the manner of the double homoclinicity (see Fig.1(b)). For general cases of odd periodic twin-chains, the merging process occurs in the same way as described above. The parameter values at which the twin-chains merge are called reconnection thresholds. The reconnection threshold for the period-one twin-chains, $K_{rec}^{(1)}$, has been derived by Howard and Hohs by using the averaged Hamiltonian method as follows [3, 4]:

$$K_{rec}^{(1)} = \frac{2}{3}(2\pi\mu)^{3/2}. \quad (5)$$

Although Eq.(5) was derived by the perturbation theory, the results explain the numerical data very well [3].

On the other hand, the twin-chains with even-periods reveal quite different features from twin-chains with odd-periods. Figure 2 is the case for the merging of the twin-chains with period-two. In the case of the even periodic twin-chains, the pair of the resonant chains appear with the same phase near the shearless KAM curve. The topological structure of the twin-chains is quite different from the odd periodic case. At a critical parameter value, the hyperbolic periodic points of the twin-chains merge (see Fig.2(b)). We will call this state the reconnection for the twin-chains with even-periods. The reconnection threshold for the period-two case, $K_{rec}^{(2)}$, has been analytically derived as [4]

$$K_{rec}^{(2)} = 2\sqrt{2\pi(\mu + \frac{1}{2})}. \quad (6)$$

3 The symmetry of the quadratic map and the shearless KAM curves

Hereafter we focus our attention on the KAM curves between twin-chains and derive the location of the shearless KAM curve. Denote the KAM curve which goes through the point $(\theta, I) = (0, I_0)$

as \mathcal{K}_{I_0} , i.e.,

$$\mathcal{K}_{I_0} = \{(\theta, I) | (\theta, I) = T^n \mathbf{x}_0, \mathbf{x}_0 = (\theta = 0, I = I_0), -\infty \leq n \leq \infty\}. \quad (7)$$

The KAM curves are characterized by the rotation number R ,

$$R = R(T, (\theta_0, I_0)) \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \frac{1}{2\pi N} \sum_{n=1}^N (\theta_n - \theta_{n-1}), \quad (8)$$

which is a function of the mapping T and initial conditions (θ_0, I_0) . By fixing the value of θ_0 , the KAM curve is parametrized by the value of I_0 .

In the quadratic map, the KAM curve remains invariant under the involutions M_1, M_2 , which is shown as follows. By using the relation $M_j T^n = T^{-n} M_j$ ($j = 1, 2$) for any integer n ,

$$M_1 T^n \mathbf{x}_0 = T^{-n} M_1 \mathbf{x}_0 = T^{-n} \mathbf{x}_0 \quad (9)$$

and

$$M_2 T^n \mathbf{x}_0 = T^{-n} M_2 \mathbf{x}_0 = T^{-n} (\theta = 2\pi\mu - I_0^2, I = I_0) = T^{-(n-1)} \mathbf{x}_0, \quad (10)$$

where $\mathbf{x}_0 = (\theta, I) = (0, I_0)$. Therefore we have

$$M_j \mathcal{K}_{I_0} = \{(\theta, I) | (\theta, I) = M_j T^n \mathbf{x}_0, \mathbf{x}_0 = (\theta = 0, I = I_0), -\infty \leq n \leq \infty\} \quad (11)$$

$$= \{(\theta, I) | (\theta, I) = T^n \mathbf{x}_0, \mathbf{x}_0 = (\theta = 0, I = I_0), -\infty \leq n \leq \infty\} \quad (12)$$

$$= \mathcal{K}_{I_0}. \quad (j = 1, 2)$$

The shearless KAM curve which passes through the point $(\theta = 0, I = I_0^*)$, say $\mathcal{K}_{I_0^*}$, is defined as the KAM curve whose rotation number R^* satisfies the condition

$$R(T, (\theta = 0, I = I_0)) \neq R^* \quad \text{for } \forall I_0 \neq I_0^*. \quad (13)$$

This definition is trivial in every part of the phase space if there exists only one shearless KAM curve in the phase space.

Moreover, it is shown that the shearless KAM curve is invariant under the transformation S as follows. By using the relation $ST^m = T^m S$, we have

$$R(T, S(\theta_0, I_0)) = R(T, (\theta_0, I_0)). \quad (14)$$

Therefore, if the shearless KAM curve goes through the point (θ_0, I_0) , it also goes through the point $S(\theta_0, I_0)$, i.e., the shearless KAM curve is an invariant set of S :

$$S\mathcal{K}_{I_0^*} = \mathcal{K}_{I_0^*}. \quad (15)$$

We require that the shearless KAM curve $\mathcal{K}_{I_0^*}$ crosses θ -axis at two points, denoted by $A : (\theta, I) = (\theta_A, 0)$ and $B : (\theta, I) = (\theta_B, 0)$ respectively. Since the shearless KAM curve is an invariant set of M_2 and S , the following two equations hold:

$$\theta_B = -\theta_A + 2\pi\mu, \quad (16)$$

$$\theta_B = \theta_A - \pi. \quad (17)$$

From the above equations, we have

$$\theta_A = \pi(\mu + \frac{1}{2}), \quad (18)$$

$$\theta_B = \pi(\mu - \frac{1}{2}), \quad (19)$$

which are K independent. Equations (18) and (19) determine the exact location of the shearless KAM curves in the phase space.

4 The breakup of the shearless KAM curves

In order to obtain the breakup threshold of shearless KAM curves, we numerically iterate the point $(\theta, I) = (\pi(\mu + 1/2), 0)$ for each set of μ and K , and examine whether the motion is bounded or not. In cases when the motion is unbounded, the shearless KAM curve is considered to have been destroyed. In our numerical calculations, the shearless KAM curve is considered to be broken up if the absolute value of I exceeds 2 during 10^5 iterations.

The numerical results are shown in Fig.3, where the motion is bounded in the black region, but the shearless KAM curve does not exist in the white region. This straightforward numerical method enables us to see the breakup condition of the shearless KAM curves. This method cannot detect the detailed threshold boundary for the breakup of the shearless KAM curves. However, the result will be much more refined by improving the breakup criterion in the numerical calculations. Actually, the breakup thresholds obtained in Fig.3 are numerically confirmed by the determination of the rotation number for the nearby KAM curves.

The breakup threshold of the shearless KAM curve characterized by the inverse golden mean $1/\gamma$ rotation number has already been determined by del-Castillo-Negrete et al. using the Greene's residue method[8]. In the case of the critical value for the destruction of the $1/\gamma$ shearless KAM curve, they obtained $\mu_c = -0.313951$ and $K_c = 1.54156$, which are indicated by the point (μ_c, K_c) in Fig.3. The shearless KAM curve for μ_c and K_c is plotted in Fig.4, which is obtained by numerically iterating the point $(\theta = \pi(\mu_c + 1/2), I = 0)$.

At the reconnection threshold of twin-chains, the rotation number of the shearless KAM curve, which exists between two twin-chains, is equal to the rotation number of the twin-chains. Thus, the reconnection threshold for the twin-chains with any period can be obtained by numerically calculating the rotation number of shearless KAM curves. In the cases of period-one and period-two, it is possible to derive the reconnection threshold analytically, as was shown previously. The P/Q lines plotted in Fig.3 are numerically determined reconnection thresholds for twin-chains with the rational rotation numbers P/Q . In the quadratic map, the rotation number of a shearless KAM curve depends on the values of both μ and K . This is because K not only represents the strength of the non-integrable perturbation[7], but also the change in the integrable part of the system.

As shown in Fig.3, the boundary of the breakup diagram reveals many sharp singularities on the lines of the reconnection thresholds with rational rotation numbers P/Q . Figure 5 shows a magnification of the breakup diagram around (μ_c, K_c) , where fine singular structures are seen. We expect that much more fine structures, corresponding to the reconnection thresholds for twin-chains with higher periods, are immersed in the breakup diagram.

5 Summary and discussion

In this paper, we have determined the exact location of the shearless KAM curve in the quadratic map. Furthermore, we succeeded in obtaining the global structure of the breakup diagram of shearless KAM curves in the entire parameter space. The breakup diagram has many sharp singularities, and they are well explained by the reconnection thresholds for twin-chains. Fine structures as shown in Fig.5 suggest that the breakup diagram might be a fractal. The theory which combines the breakup of shearless KAM curves with the reconnection of twin-chains remains to be established.

As mentioned by del-Castillo-Negrete et al., a shearless KAM curve exhibits self-similar hierarchical structures for certain parameter values[8]. Now that the exact location of a shearless KAM curve is obtained, it can be used to study the geometry of shearless KAM curves. The detailed analysis of the breakup diagram and the geometrical properties of shearless KAM

curves will be reported elsewhere.

References

- [1] J. Moser, Nachr. Akad. Wiss. Göttingen, Math. -phys. **K1**,11a,No.1(1962), 1.
- [2] J.D. Meiss, Rev. Mod. Phys. **64** (1992), 795.
- [3] J.E. Howard and S.M. Hohns, Phys. Rev. **A29** (1984), 418.
- [4] J.E. Howard and J. Humpherys, Physica **D80** (1995), 256.
- [5] J.P. Van der Weele, T.P. Valkering, H.W. Capel and T. Post, Physica **A153** (1988), 283.
- [6] J.P. Van der Weele and T.P. Valkering, Physica **A169** (1990), 42.
- [7] R. Egydio de Carvalho and A.M. Ozorio de Almeida, Phys.Lett.**A162** (1992), 457.
- [8] D. del-Castillo-Negrete, J.M. Greene and P.J. Morrison, Physica **D91** (1996), 1.

Figure captions

Fig.1. The merging process of the twin-chains with period-one for various values of K at $\mu = 0.05$. (a) $K=0.05$ (below threshold). (b) $K=0.1173906615$ (reconnection threshold). (c) $K=0.20$ (above threshold).

Fig.2. The merging process of the twin-chains with period-two for various values of K at $\mu = -0.49$. (a) $K=0.4$ (below threshold). (b) $K=0.50132565$ (reconnection threshold). (c) $K=0.7$ (above threshold).

Fig.3. The breakup diagram of shearless KAM curves and the reconnection thresholds for the twin-chains with rational rotation numbers P/Q . The critical point ($\mu_c = -0.313951, K_c = 1.54156$) obtained by del-Castillo-Negrete, Greene and Morrison is indicated by an arrow.

Fig.4. The shearless KAM curve for (μ_c, K_c) .

Fig.5. A magnification of the breakup diagram around (μ_c, K_c) .

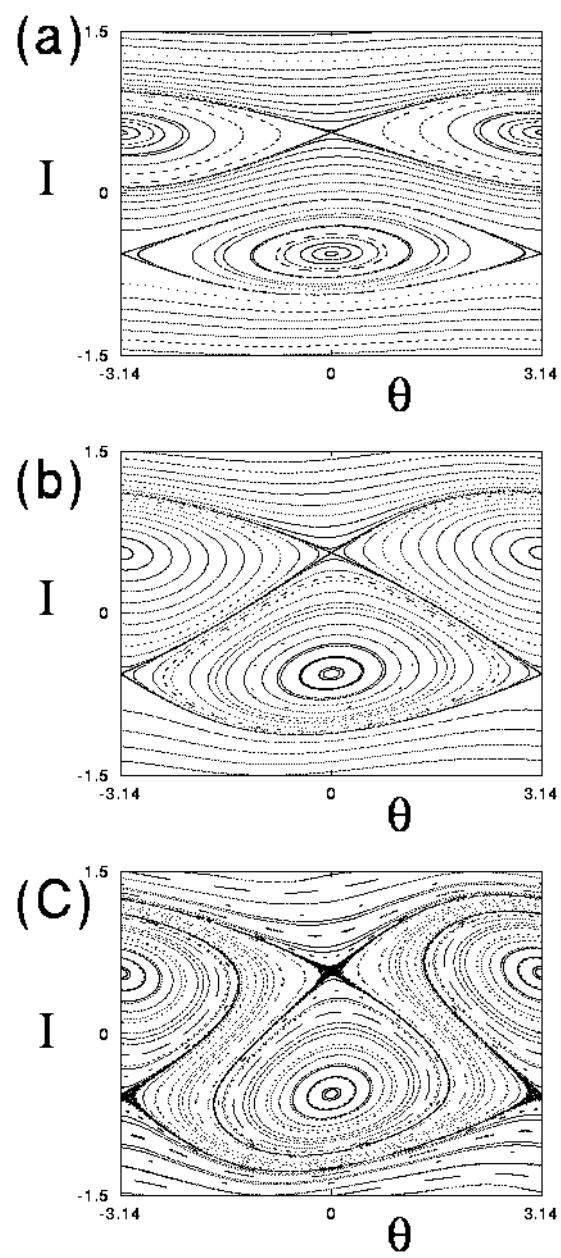


Fig.1

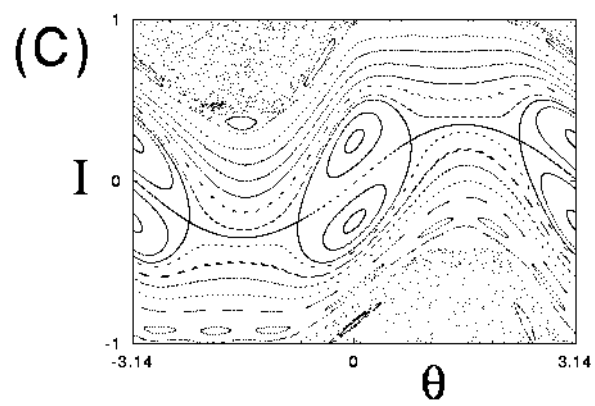
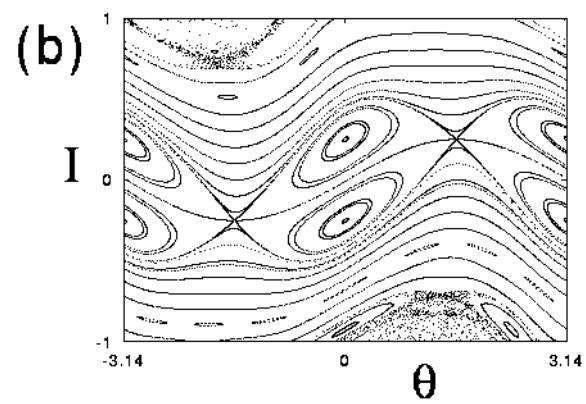
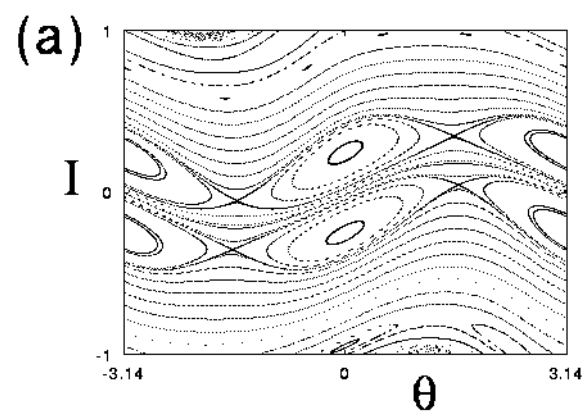


Fig.2

This figure "SA_Fig3.gif" is available in "gif" format from:

<http://arXiv.org/ps/chao-dyn/9703011v1>

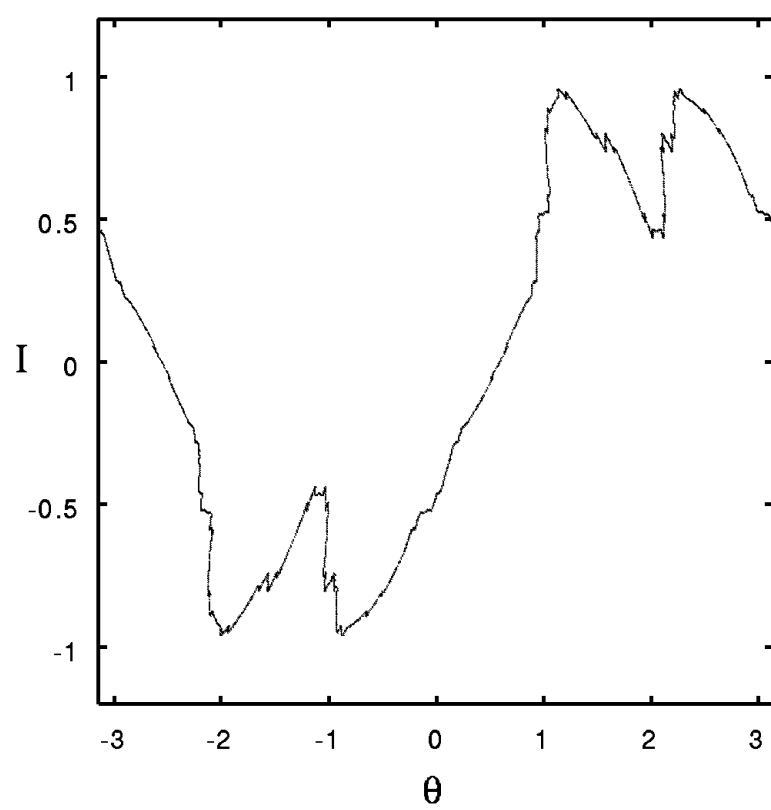


Fig.4

This figure "SA_Fig5.gif" is available in "gif" format from:

<http://arXiv.org/ps/chao-dyn/9703011v1>